

The nonlinear aerodynamic stability of long-span bridges: post flutter

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ABSTRACT: The linear flutter analytical method can't give the reasonable explanation to post flutter status of bridges. Based on the nonlinear MIAF expressions, the nonlinear vibration differential equation can describe the aerodynamic stability of bridge under different amplitudes and reduced frequency. The nonlinear aerodynamic coefficients in the expression can be represented by the amplitude and initial phase of harmonic components. The nonlinear aerodynamic stability analysis had been performed in state space using the 4th Runge-Kutta algorithm, and the results indicated that the girder of long span bridge would appear different motion types in post flutter, including the convergence motion, the divergence motion and the identical amplitude oscillation. The girder of long span bridge can keep the identical amplitude oscillation status at different amplitude conditions, and the only requirement is the equilibrium in the aerodynamic works.

KEYWORDS: post flutter; nonlinear aerodynamic coefficient; nonlinear aerodynamic stability

1 INTRODUCTION

The motion-induced aerodynamic force (MIAF) in Scanlan's expression is only the function of reduced frequency, and is independent with the amplitude of bridge girder, and there are no nonlinear items and variables in the expression. According to the linear stability theory, there is only a critical status in flutter equation, thus it can't give the further results about aerodynamic stability of bridges especially in the post flutter status. Indeed, the large amplitude oscillation of Tacoma Bridge is typical motion of post flutter, although its collapse was only explained for flutter. The stability status or motion status in post flutter of long span bridges needs to be investigated by the means of nonlinear differential equation considering the nonlinear MIAF.

For the issue of large amplitude oscillation of Tacoma Bridge, several researchers analyzed the aerodynamic mechanism based on nonlinear aerodynamic force and nonlinear vibration theory. Steinmann^[2] proposed an aerodynamic force expression of Tacoma deck in 1954, then Böhm^[1] studied the nonlinear oscillation of Tacoma Bridge, and interpreted that the large amplitude oscillation of Tacoma Bridge was a limit cycle oscillation. Piccardo^[3] also studied the nonlinear vibration of Tacoma Bridge by using Steinmann's expression, and obtained the wind speed 17.44m/s under the limit cycle oscillation of this bridge. Xu^[4,5] deduced an expression of nonlinear aerodynamic force and nonlinear vibration equation, and obtained the nearly same critical wind speed as Piccardo under limit cycle oscillation state. Due to the restriction of wind tunnel test technique in acquiring nonlinear aerodynamic force, other bridges are scarcely studied on its nonlinear aerodynamic force or the post flutter state. Diana^[6-8] studied nonlinear aerodynamic force of Messina Bridge girder, and indicated that the neglect of nonlinear component of aerodynamic force could lead to unsafely side for aerodynamic instability. Liao^[9] found the distinct 8-shape aerodynamic hysteresis loops of streamline box

girder under the larger pitching amplitude, which means that the aerodynamic forces can provide both the positive and negative works within one vibration period. And the 8-shape aerodynamic hysteresis loops also gave a favorable support of limit cycle oscillation of long-span bridge. Chen^[10] indicated that the soft flutter phenomenon of long-span bridge was a typical nonlinear vibration, and he used Von Del Pol equation to describe the phenomenon and the nonlinear MIAF of bridge girder.

The other researches on nonlinear aerodynamic oscillation of flexible and long-span structures, were mainly focused on the galloping, such as iced transmission line. Den Hartog^[11,12] studied the galloping of iced transmission line in details first time, and proposed the analysis method of galloping using semi-steady theory. Parkinson^[13] studied the galloping theory of square column in details. He studied the process of square column galloping by aero-elastic model wind tunnel tests, and found the leap and hysteresis phenomenon in galloping under different reduced frequency. Novak^[14], based on SDOF torsional vibration model, studied the galloping of square column in details. His theoretical researches can be correspondent with the wind tunnel test results conducted by Parkinson. The galloping theory based on the semi-steady has developed greatly, the main reason is the simplification of nonlinear aerodynamic force. Lift and pitching moment acting on a transmission line, or square column, has been described only as the function of incidence angle, namely the aerodynamic force can be expressed only with incidence angle.

Although the nonlinear aerodynamic force of bridge girder can be described at very low reduced frequency, under which the aerodynamic hysteresis loop is almost a single curve (this phenomenon was verified in wind tunnel test by Diana^[6] and Liao^[9]), the bridge may occur flutter at a common reduced frequency, under which the aerodynamic hysteresis is loop.

Combining with the nonlinear MIAF expressions^[15], the nonlinear vibration differential equation can describe the aerodynamic stability of bridge girder under different amplitudes and reduced frequency, especially the large amplitude oscillation in the post flutter status of bridges. The nonlinear aerodynamic stability analysis had been performed in state space by using the 4th Runge-Kutta algorithm, and the results indicated that the girder of long span bridge would appear different motion types in post flutter, including the convergence motion, the divergence motion and the identical amplitude oscillation (limit cycle oscillation). The girder of long span bridge can keep the identical amplitude oscillation status at different amplitude conditions, and the only requirement is the equilibrium in the aerodynamic works.

2 AERODYNAMIC FORCE AND COEFFICIENTS

2.1 Typical aerodynamic force

Based on the results of the wind tunnel tests, the empirical mathematic model^[15] was proposed to describe the nonlinear MIAF under different conditions by using Taylor expanding. Before the analysis, the typical aerodynamic forces under different conditions were selected from the reference paper^[16], and one of them represented the special energy inputting to the girder. The outline of the deck cross section is shown in Fig.1, and the wind tunnel test was interpreted in the reference paper^[9] and^[16]. The selected aerodynamic forces were presented in the form of hysteresis loop, and all of them with the same static incidence angle +5 degree. The forces were shown in Fig.2 ~ Fig.5 respectively and the illustration were as follows.

The aerodynamic force in Fig.2 was obtained under the condition of 10° amplitude and $\pi/5$ reduced frequency, and defined as the case-1. The direction of rotation is counter clock,

which indicates the aerodynamic work is negative. The aerodynamic force in Fig.3 was obtained under the condition of 10° amplitude and $\pi/8$ reduced frequency, and defined as the case-2. Varying with the decrease of reduced frequency, the hysteresis loop became the 8-shape, which means that the aerodynamic forces can provide both the positive and negative works within one vibration period. The area of clockwise is less than the counter-clockwise, namely the negative one is greater than the positive one a little. The aerodynamic force in Fig.4 was obtained under the condition of 10° amplitude and $\pi/10$ reduced frequency, and defined the case-3, and the negative work is less than the positive work a little while the K decreasing. The aerodynamic force in Fig.5 was obtained under the condition of 20° amplitude and $\pi/10$ reduced frequency, and defined the case-4, and positive one is more greater than the negative one while the amplitude increasing.

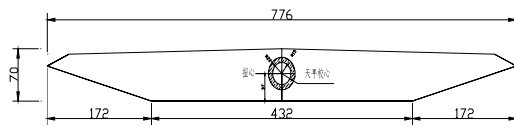


Fig. 1. The cross-section of test model

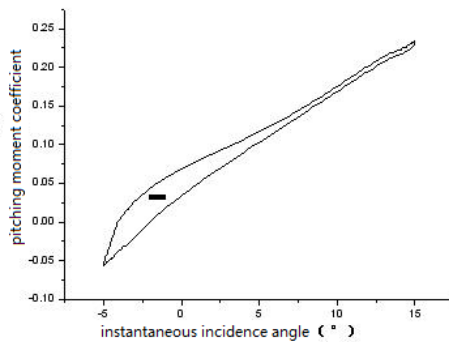


Fig.2 the hysteresis loop of pitching moment coefficient (case-1: $K=\pi/5$, Amplitude=10degree)

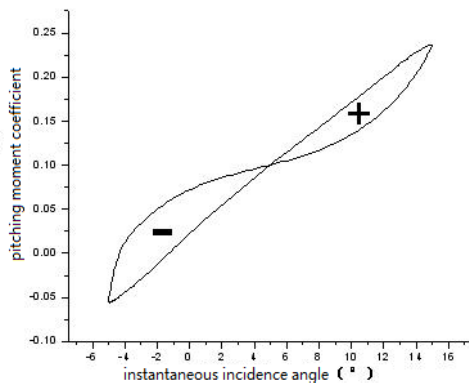


Fig.3 the hysteresis loop of pitching moment coefficient (case-2: $K=\pi/8$, Amplitude=10degree)

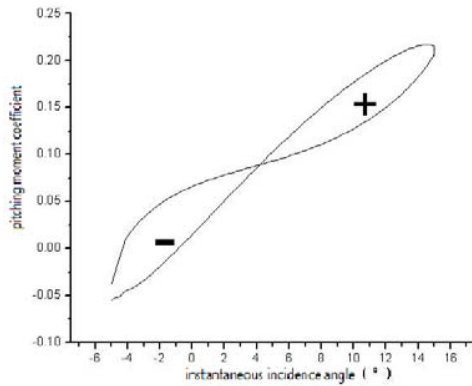


Fig.4 the hysteresis loop of pitching moment coefficient (case-3: $K=\pi/10$, Amplitude=10degree)

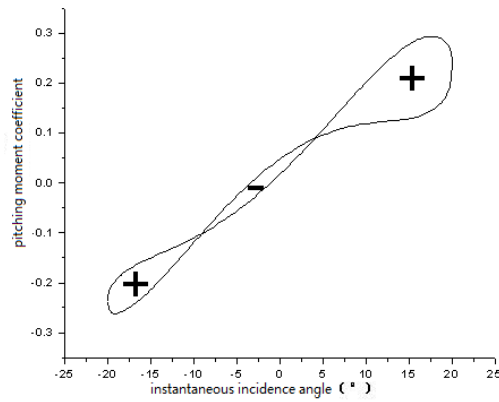


Fig.5 the hysteresis loop of pitching moment coefficient (case-4: $K=\pi/10$, Amplitude=20degree)

2.2 Aerodynamic coefficient

The aerodynamic coefficients were obtained by the method proposed in reference paper^[15], and expressed as follows.

$$C_{\dot{\alpha}^m \alpha^n}^k = \frac{X_{TM}^k k!}{\omega^m} \sin(\theta_{TM}^k + \frac{n\pi}{2}) \quad (1)$$

In which, $k=1,2,\dots,n$, is the order of nonlinear harmonic components; m is the nonlinear aerodynamic coefficient order which represents the derivative order of velocity and displacement. X_{VL}^k is the pitching amplitude of the k^{th} order torsional pitching moment harmonic component and θ_{VL}^k is the initial phase angle.

If taking the first five orders in aerodynamic coefficient, the C_M can be written as,

$$\begin{aligned} C_M = & C_{\dot{\alpha}}^1 \dot{\alpha} + C_{\alpha}^1 \alpha \\ & + \frac{1}{2} C_{\dot{\alpha}^2}^2 \dot{\alpha}^2 + C_{\alpha \dot{\alpha}}^2 \dot{\alpha} \alpha + \frac{1}{2} C_{\alpha^2}^2 \alpha^2 \\ & + \frac{1}{6} C_{\dot{\alpha}^3}^3 \dot{\alpha}^3 + \frac{1}{2} C_{\alpha \dot{\alpha}^2}^3 \dot{\alpha}^2 \alpha + \frac{1}{2} C_{\alpha^2 \dot{\alpha}}^3 \dot{\alpha} \alpha^2 + \frac{1}{6} C_{\alpha^3}^3 \alpha^3 \\ & + \frac{1}{24} C_{\dot{\alpha}^4}^4 \dot{\alpha}^4 + \frac{1}{6} C_{\alpha \dot{\alpha}^3}^4 \dot{\alpha}^3 \alpha + \frac{1}{4} C_{\alpha^2 \dot{\alpha}^2}^4 \dot{\alpha}^2 \alpha^2 + \frac{1}{6} C_{\alpha^3 \dot{\alpha}}^4 \dot{\alpha} \alpha^3 + \frac{1}{24} C_{\alpha^4}^4 \alpha^4 \\ & + \frac{1}{120} C_{\dot{\alpha}^5}^5 \dot{\alpha}^5 + \frac{1}{24} C_{\alpha \dot{\alpha}^4}^5 \dot{\alpha}^4 \alpha + \frac{1}{12} C_{\alpha^2 \dot{\alpha}^3}^5 \dot{\alpha}^3 \alpha^2 + \frac{1}{12} C_{\alpha^3 \dot{\alpha}^2}^5 \dot{\alpha}^2 \alpha^3 + \frac{1}{24} C_{\alpha^4 \dot{\alpha}}^5 \dot{\alpha} \alpha^4 + \frac{1}{120} C_{\alpha^5}^5 \alpha^5 \end{aligned} \quad (2)$$

In order to show the coefficients clearly, Table 1 gives the correspondence of the coefficients and characters. The coefficients of aerodynamic force shown in Figs. 2-5 were illustrated in Tables 2-5.

Table 1 Correspondence of the coefficients and characters

	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$
$C1$	$C_{\dot{\alpha}}^1$	C_{α}^1				
$C2$	$C_{\dot{\alpha}^2}^2$	$C_{\alpha\dot{\alpha}}^2$	$C_{\alpha^2}^2$			
$C3$	$C_{\dot{\alpha}^3}^3$	$C_{\alpha\dot{\alpha}^2}^3$	$C_{\alpha^2\dot{\alpha}}^3$	$C_{\alpha^3}^3$		
$C4$	$C_{\dot{\alpha}^4}^4$	$C_{\alpha\dot{\alpha}^3}^4$	$C_{\alpha^2\dot{\alpha}^2}^4$	$C_{\alpha^3\dot{\alpha}}^4$	$C_{\alpha^4}^4$	
$C5$	$C_{\dot{\alpha}^5}^5$	$C_{\alpha\dot{\alpha}^4}^5$	$C_{\alpha^2\dot{\alpha}^3}^5$	$C_{\alpha^3\dot{\alpha}^2}^5$	$C_{\alpha^4\dot{\alpha}}^5$	$C_{\alpha^5}^5$

Table 2 Aerodynamic coefficients for case-1

	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$
$C1$	-0.0061	0.7576	0.0000	0.0000	0.0000	0.0000
$C2$	0.0041	0.0500	-0.6016	0.0000	0.0000	0.0000
$C3$	0.0039	-0.0473	-0.5667	6.9572	0.0000	0.0000
$C4$	-0.0058	-0.0144	0.8540	2.1116	-125.5797	0.0000
$C5$	-0.0015	0.0370	0.2227	-5.4456	-32.7507	800.7876

Table 3 Aerodynamic coefficients for case-12

	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$
$C1$	-0.0034	0.7311	0.0000	0.0000	0.0000	0.0000
$C2$	0.0007	0.1651	-0.0701	0.0000	0.0000	0.0000
$C3$	0.0066	-0.1245	-0.6803	12.7382	0.0000	0.0000
$C4$	-0.0028	-0.0144	0.2916	1.4759	-29.8426	0.0000
$C5$	-0.0111	0.0737	1.1382	-7.5422	-116.4733	771.8068

Table 4 Aerodynamic coefficients for case-3

	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$
$C1$	0.0029	0.6873	0.0000	0.0000	0.0000	0.0000
$C2$	0.0044	0.2381	-0.2920	0.0000	0.0000	0.0000
$C3$	0.0085	-0.0772	-0.5607	5.0724	0.0000	0.0000
$C4$	-0.0088	-0.0232	0.5804	1.5230	-38.1319	0.0000
$C5$	0.0136	0.2087	-0.8938	-13.7102	58.7196	900.7039

Table 5 Aerodynamic coefficients for case-4

	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$
$C1$	0.0081	0.7103	0.0000	0.0000	0.0000	0.0000
$C2$	0.0041	0.0450	-0.2679	0.0000	0.0000	0.0000
$C3$	-0.0118	0.0228	0.7776	-1.5007	0.0000	0.0000
$C4$	0.0007	0.0004	-0.0459	-0.0270	3.0179	0.0000

C5	0.0014	0.0070	-0.0934	-0.4621	6.1378	30.3609
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3 NONLINEAR DIFFERENTIAL EQUATION AND SOLUTION

3.1 Nonlinear differential equation.

Based on the direct Taylor expanding of nonlinear MIAF, and combined with the free vibration equation of the dynamic system, the nonlinear differential equation was constructed to describe the nonlinear aerodynamic stability of bridge girder. Because of the nonlinear items and variables in this equation, the limit cycle, the convergence motion and the divergence motion may exist under a special condition, which can prediction the motion of long bridges at the post-flutter status.

Taking an example of aerodynamic moment under torsional motion condition, the nonlinear differential Eq. can be written as follows:

$$\ddot{\alpha} + (2\xi_{\alpha}\omega_{\alpha} - \eta)\dot{\alpha} + (\omega_{\alpha}^2 - \mu)\alpha = 0 \quad (3)$$

In above, $\eta = f(\dot{\alpha}, \alpha)$, $\mu = g(\dot{\alpha}, \alpha)$, and both of them are the nonlinear function of motion variables of bridge girder. Considering the motion variables (velocity and displacement) in the wind tunnel tests are certain values, and the parameter η and μ is also a certain value, and the solution of Eq. 3 in the different case only represents the tendency of girder motion with a special condition (including the dynamic parameters and initial motion parameters).

Because the 2nd order variables in Eq. 3 could bring on the difficult in solving, the state variables, which can reduce the order of the variables, are introduced, thus the equation can be transformed to the 1st order in the state space,

$$\dot{Y} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \dot{\alpha} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} y_2 \\ -(2\xi_{\alpha}\omega_{\alpha} - \eta)y_2 - (\omega^2 - \mu)y_1 \end{bmatrix} = \begin{bmatrix} \dot{\alpha} \\ -(2\xi_{\alpha}\omega_{\alpha} - \eta)\dot{\alpha} - (\omega^2 - \mu)\alpha \end{bmatrix} \quad (4)$$

$$Y_0 = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \dot{\alpha}_0 \end{bmatrix} \quad (5)$$

Giving the initial condition, the Eq. 3 can be solved using the 4th Runge-Kutta algorithm, and the results show the locus of the motion, which represent the motion status in post flutter.

3.2 Solving conditions

The general nonlinear differential equation to investigate the nonlinear aerodynamic stability of bridge girder under the special condition is shown below,

$$\ddot{\alpha} + 2\xi\omega\dot{\alpha} + \omega^2\alpha = pC_M^T(\alpha, \dot{\alpha}) \quad (6)$$

where $p = \frac{1}{2}\rho U^2 B^2 L / I_m$ is the merging term, and the ρ =air density, U = wind speed, B = girder width, L = girder length, I_m = mass moment of inertia, ξ =damping ratio, ω =cycle frequency. $C_M^T(\alpha, \dot{\alpha})$ is the pitching moment coefficient of the different cases illustrated in Tables 2-5.

The standard form of solution in state space is as follow,

$$\dot{Y} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \dot{\alpha} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} y_2 \\ -2\xi\omega y_2 - \omega^2 y_1 - pC_M^T \end{bmatrix} = \begin{bmatrix} \dot{\alpha} \\ -2\xi\omega\dot{\alpha} - \omega^2\alpha - pC_M^T \end{bmatrix} \quad (7)$$

The initial condition in solution is ,

$$\alpha_0=0, \quad \dot{\alpha}_0=0.1745\omega$$

The width of the girder in analysis is 0.776m, and considering the 2D analysis assumption the length is 1.0m. The other initial parameters for 4 cases are listed in Table 6.

Table 6 Parameters in different cases

	$\omega(\text{rad/s})$	$2\xi\omega$	$U(\text{m/s})$	$I_m(\text{Kgm}^2)$	$p(/s^2)$
Case1	12.13	0.01	15	1.0	82.987
Case2	10.12	0.01	20	2.0	73.766
Case3	8.11	0.01	20	2.5	59.013
Case4	8.11	0.01	20	8.0	18.442

4 NONLINEAR AERODYNAMIC STABILITY OF POST FLUTTER

The different motion corresponding to the differential equation can be described in solution curve and phase plane diagram. Generally, the analysis results can express as follows: the positive work in the hysteresis loop is less than the negative one, the motion of bridge girder will converge, see in Fig.6 and Fig.7. On the other hands, the motion will converge while the aerodynamic positive work is greater than the negative one, see in Fig.8. The identical amplitude oscillation could appear while the energy which structure consumed is equal to the aerodynamic energy, as shown in Fig.9. Because the area of positive work is almost equal to negative work in case-3, the speed of convergence is slow down dramatically, and the amplitude trends to the identical. The larger mass moment can consume the more aerodynamic energy in one period, and the identical amplitude oscillation can also occur even if the positive aerodynamic work is larger than the negative one. For case-4, if the I_m become the twice of the original one, and the amplitude will be identical, and the girder could occur limit cycle oscillation, see Fig.10

On the contrary, if the structural stiffness is so small, or the mass moment is too small to consume the aerodynamic energy in one period, the displacement of bridge girder increases vary rapidly, and the aerodynamic negative damping and structural damping can't consume the positive energy in time, even if the negative work is larger than the positive one, the girder motion will divergence directly as show in Fig. 11. In fact, because of the exist of 8-shape loop under different conditions, the motion of bridges girder may tend to the limit cycle oscillation, and may convert to different amplitude depending on wind speed. Of course, the girder could also occur the direct divergence while the positive aerodynamic energy increasing.

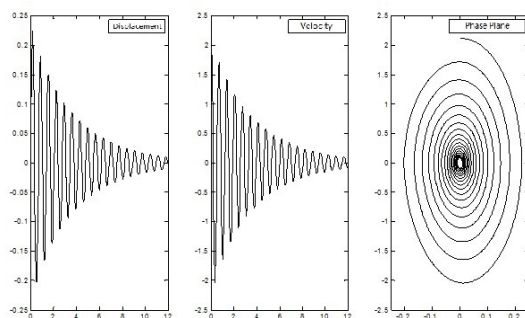


Fig.6 The solution curve and the phase plane diagram (case-1)

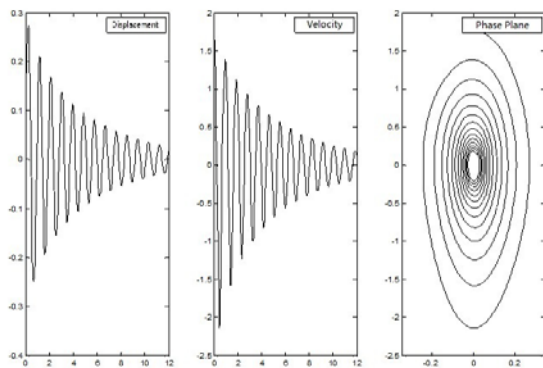


Fig.7 The solution curve and the phase plane diagram (case-2)

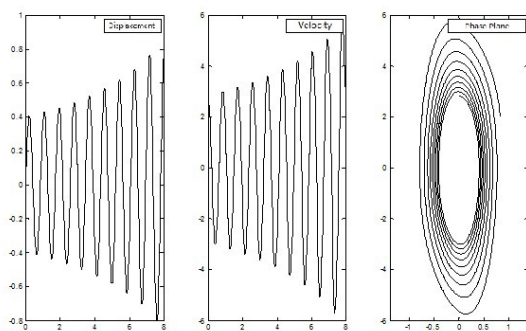


Fig.8 The solution curve and the phase plane diagram (case-4)

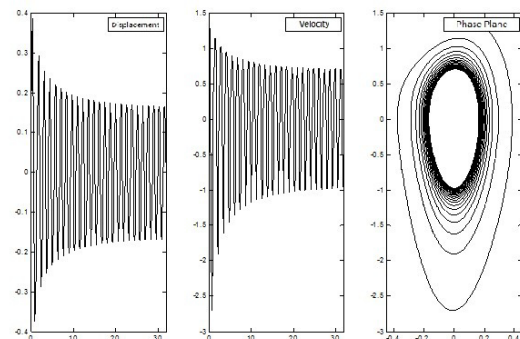


Fig.9 The solution curve and the phase plane diagram (case-3)

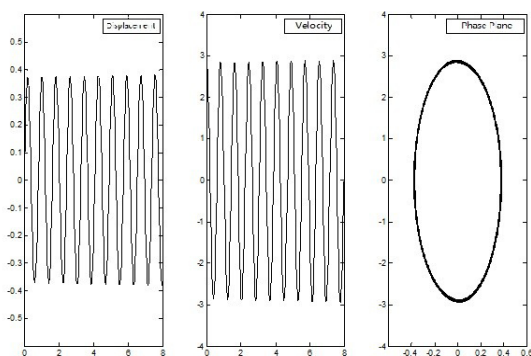


Fig.10 The solution curve and the phase plane diagram (case-4, $I_m=16$)

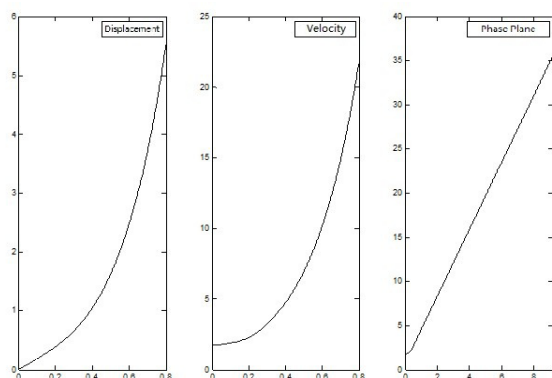


Fig.10 The solution curve and the phase plane diagram (case-2, $I_m=1$)

5 CONCLUSIONS

By introducing nonlinear MIAF expressions, nonlinear vibration differential equation can describe the aerodynamic stability of bridge girder under different amplitudes and reduced frequency, especially the large amplitude oscillation in the post flutter status of bridges. The nonlinear aerodynamic stability analysis had been solved in state space by using the 4th Runge-Kutta algorithm, and results show that the girder of long span bridge would appear different motion types in post flutter, including the convergence motion, the divergence motion and the identical amplitude oscillation (limit cycle oscillation). The motion of bridges girder may tend to the limit cycle oscillation, and convert to different amplitude due to 8-shape loop under different conditions. The girder can also occur the direct divergence while the positive aerodynamic energy increasing.

6 ACKNOWLEDGEMENTS

The paper is based on the project: The study on nonlinear motion-induced aerodynamic force and nonlinear aerodynamic stability of long-span bridge, which is supported by the Major Research plan of the National Natural Science Foundation of China (Grant No. 90815016).

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