Vortex-induced vibration prediction of long-span bridges considering imperfect correlation of aerodynamic forces

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ABSTRACT: Periodical wake vortices may cause vortex induced vibration (VIV) of a long span bridge, the vortex-induced aerodynamic forces produced by vortices are partial correlation along the span. Based on Scanlan’s semi-empirical linear and nonlinear models, the correlation of vortex-induced aerodynamic forces along span is studied respectively in frequency domain, and reduced factor of vortex-induced response between two-dimensional model to three-dimensional prototype structure is defined, and the methodology for extending the test results of section model into prototype structure by considering partial correlation of vortex-induced aerodynamic force along span is discussed. Validity of proposed theory is proved by sectional model tests and observed results in field.

KEYWORDS: Vortex-induced vibration; long span bridge; linear and nonlinear force models; partial correlation; wind tunnel testing.

1 INTRODUCTION

Vortex-induced vibration (VIV) of line-like structures, such as flexible long-span bridge, is one of the most common aeroelastic phenomena due to wind-structure interaction at relative low wind speeds. In the recent decades, obvious vortex-induced vibration has been observed in many long-span bridges (Kumarasena 1991, Larsen 2000, Fujino 2002). Various mathematical models that reflect the main aspects of the vortex-induced response of bluff bodies have been proposed (Hartlen & Currie, 1970, Iwan & Blevins, 1974, Scanlan, 1981, Larsen 1995), and most of these models were developed for the 2-D model. Elastically-mounted, rigid, section model (2-D) wind tunnel testing technique becomes a general methodology to investigate VIV of long-span bridges. However, the vortex-induced aerodynamic force produced by vortices is imperfect correlation along the span (Williamson 1988, Miller 1994). Consistence between the observation of prototype bridge and predictions based on 2-D model testing is still uncertain due to contribution of spatial spanwise correlation of vortex-induced aerodynamic forces (Wilkinson 1981). There is still lack of comprehensive theory to extend the test results of section into prototype bridge vibration because of the fact that the existence of mode shapes and the nonlinear vortex-induced forces are not fully correlated along the span, although several researchers tried to study it (Ehsan and Scanlan 1990, Zhu 2005, Xian 2008).

Two methods for considering the effect of imperfect span-wise correlation of aerodynamic forces on the vortex-induced response of long-span flexible bridges are described in this paper. These methods are based on semi-empirical linear and nonlinear mathematical model respectively. Validity of proposed theory is proved by sectional model tests and observed results in field.
2 LINEAR METHOD

2.1 Linear model description

By considering the existence of phase-difference along the span, the across-flow vortex-induced vibration of a rigid bluff body based on Scanlan’s semi-empirical linear force model (1981) should take the form (here take the lift as an example):

\[ m(\ddot{y} + 2\zeta\omega_0 \dot{y} + \omega_0^2 y) = \frac{1}{2} \rho U^2 D [Y_1(K) \frac{\dot{y}}{U} + Y_2(K) \frac{y}{D} + \frac{1}{2} C_L(K) \sin(\alpha t + \psi(x))] \]  

(1)

Where, \( m \)=mass of per unit span length; \( \zeta \)=damping ratio-to-critical; \( \omega_0 \)=mechanical circular frequency; \( \rho \)=air density; \( U \)=the velocity of wind flow; \( D \)=across-flow dimension of section; \( y(x,t) \)=the vortex-induced vibration of the body, \( x \)=along-span degree-of-freedom; \( K=\omega D/U \), \( \omega \) is vortex shedding frequency that satisfies the Strouhal relation \( 2\pi S=\omega D/U \) (\( S \) is Strouhal number) ; \( Y_1(K), Y_2(K), C_L(K) \) and \( \psi(x) \) are unknown parameters to be identified by experiments. The vertical motion \( y \) at the location \( x \) along the deck can be expressed as:

\[ y(x,t) = \phi(x)\xi(t)D \]  

(2)

In which, \( \phi(x) \)=mode shape; \( \xi \)=generalized coordinate. The dimensionless form of the mathematical model can be written as the following equation by inserting Eq.2 into Eq.1, multiplying both sides by \( \phi(x) \) and integrating over the span length \( L \),

\[ \ddot{\xi}(s) + (2\zeta K_0 - \frac{\rho D^2}{2m_{eq}} Y_1) \dot{\xi}(s) + (K_0^2 - \frac{\rho D^2}{2m_{eq}} Y_2) \xi(s) = \frac{\Gamma F(s)}{\varrho} \]  

(3)

Here,

\[ s = \frac{Ut}{D}; \quad \Gamma = \frac{\rho D^2 C_L}{4m_{eq}}; \quad \varrho = \frac{1}{L} \int_{L/2}^{L/2} \phi^2(x)dx; \quad m_q = \frac{M}{L\varrho}; \quad M = \int_{L/2}^{L/2} m(x)\phi(x)\dot{\phi}(x)dx; \]

\[ F(s) = \frac{1}{L} \int_{L/2}^{L/2} |\phi(x)| \sin[Ks + \psi(x)]dx \]

The correlation of \( F(s) \) along the span is studied in following. \( F(s) \) can be written in the form:

\[ F(s) = \frac{1}{L} \int_{L/2}^{L/2} |\phi(x)| f(s,x)dx \]  

(4)

Where, \( f(s,x) = \sin[Ks + \psi(x)] \), the correlation of \( F(s) \) can be expressed as:

\[ R_{F(t)} = \frac{1}{L^2} \int_{L/2}^{L/2} \int_{L/2}^{L/2} |\phi(x_1)| |\phi(x_2)| R_{f(t)}dx_1dx_2 \]  

(5)

Fourier transform is applied in both sides of above equation:

\[ S_x(\omega) = \frac{1}{L} \int_{L/2}^{L/2} \int_{L/2}^{L/2} |\phi(x_1)| |\phi(x_2)| S_{f(\omega)}(\omega, |x_2 - x_1|)dx_1dx_2 \]  

(6)

Correlation function of three-dimensional spectrum can be written as (Houblt 1958):

\[ R(M, M', \omega) = S_x(M, M', \omega) / S(\omega) \]  

(7)
Here, $M$ and $M'$ represent any two points in 3-D space, Eq.6 can be rewritten as follow by using Eq.7,

$$S_F(\omega) = \frac{1}{L^2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \left| \phi(x_1) \right| \cdot \left| \phi(x_2) \right| \cdot R(\omega, |x_2 - x_1|) S(\omega) dx_1 dx_2$$

(8)

Where $S(\omega)$=power spectrum of vortex induced force at any cross section of span; correlation function $R$ is independent with frequency $\omega$, as vortex induced vibration is always one participation mode at lock-in. Eq.8 can be written as follow:

$$S_F(\omega) = R_F S(\omega)$$

(9)

$$R_F = \frac{1}{L^2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \left| \phi(x_1) \right| \cdot \left| \phi(x_2) \right| \cdot R(|x_2 - x_1|) dx_1 dx_2$$

(10)

Auto-convolution Integral of mode shape is pre-defined as following form (Li and Tanaka 1996):

$$\theta(\Delta x) = 2 \int_{-L/2}^{L/2} \left| \phi(x) \right| \cdot \left| \phi(x + \Delta x) \right| dx$$

(11)

The reduced relation of vortex induced forces between two-dimensional and three-dimensional is $F = \Phi f$. After inserting Eq.11 into Eq.10, and extracting square root of Eq.9. Reduced factor $\Phi$, representing the transformation of vortex induced response between two-dimensional and three-dimensional, is defined as,

$$\Phi = \frac{1}{L} \sqrt{\int_{0}^{L} \theta(\Delta x) R(\Delta x) d\Delta x}$$

(12)

Where $R(\Delta x)$ is correlation function of vortex-induced aerodynamic force along span, $\Delta x$ is spacing along span. It is obvious that reduce factor belongs to the range of $(0,1]$.

The Eq.3 can be written in the following form,

$$\ddot{\xi}(s) + (2\zeta K_0 - \frac{\rho D^2}{2m_{eq}} Y_1) \dot{\xi}(s) + \left( K_0^2 - \frac{\rho D^2}{2m_{eq}} Y_2 \right) \xi(s) = \frac{\Gamma \Phi \sin[K s + \psi_0]}{\delta}$$

(13)

The simplified form of Eq.13 can be rewritten as,

$$\ddot{\xi}(s) + 2\zeta \hat{K}_0 \dot{\xi}(s) + \hat{K}_0^2 \xi(s) = \frac{\Gamma \Phi}{\delta} \sin[K s + \psi_0]$$

(14)

In which,

$$2\zeta \hat{K}_0 = 2\zeta K_0 - \frac{\rho D^2}{2m_{eq}} Y_1; \ \ \hat{K}_0^2 = K_0^2 - \frac{\rho D^2}{2m_{eq}} Y_2.$$

The amplitude of the generalized coordinate can be calculated by:

$$\xi(s) = \frac{\Gamma \Phi}{\delta \hat{K}_0^2} \sin[K s - \theta] \theta = \arctan \left( \frac{2\beta \beta^*}{1 - \beta^2} \right) \ \ \left( \beta = \frac{K}{\hat{K}_0} \right)$$

(15)
2.2 Application of linear method

As there is no correlation function available for the concerned bridge, the correlation function proposed by Wilkinson (1981) based on pressure measurement on square cylinder is used in present analysis (Fig. 1):

\[ R(\Delta x) = \exp\left(-f_1(\eta)\left(\frac{\Delta x}{D}\right)^{f_2(\eta)}\right) \]  

(16)

Here, \( \eta \) is non-dimensional amplitude, \( f_1(\eta) \) and \( f_2(\eta) \) are fitted as the following functions (Ehsan 1990):

\[ f_1(\eta) = \frac{0.052}{0.298 + \eta^{0.25}} \quad f_2(\eta) = \frac{0.065}{0.042 + \eta} \]  

(17)

For section model case, the reduced factor varies with length of model and amplitude. The mode shape function is \( \phi(x) = 1 \), \( \theta(\Delta x) \) and \( R(\cdot) \) can be calculated by Eqs.11 and 16 respectively. The reduced factor \( \Phi \) is then obtained from Eq.12. Fig. 2 shows the reduced factors of section model varying with length of model and amplitude. For prototype or full bridge model of long span suspension bridge, the mode shape function of first symmetric vertical (V-S-1) mode can be simplified as \( \phi(x) = \cos(\pi x/L) \). The reduced factor can also be obtained from above mentioned equations. Fig. 3 shows the reduced factors for prototype bridge or full bridge model varying with length and amplitude (V-S-1).

The amplitude derived from section model testing can be interpreted into prototype bridge by the following iteration equation:

\[ A_{m+1} = n A_m \cdot \frac{\Phi_m \Theta_m}{\Phi_p \Theta_p} \]  

(18)

In above, \( n \) is model scale, \( A \) is amplitude, \( m \) and \( p \) represent model and prototype.

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Fig. 1 Examples of correlation curves (Wilkinson, 1981)
The maximum amplitude of prototype bridge can be found by Eq. 16 after several iteration. Special attention is required that the amending for mode shape has been finished in the first iterative process, and it shouldn’t be included in the rest iterative processes. It means that the iterative equation of the rest processes can be written as

\[ A_{p i+1} = nA_m \Phi_{p i} / \Phi_m \]

3 NONLINEAR METHOD

3.1 Nonlinear model description

Scanlan (1981) proposed a semi-empirical nonlinear model. In this model, a linear mechanical oscillator is subjected to a forcing function which is assumed to be the sum of a motion-induced component and a direct vortex shedding component:
The parameters $Y_1, \varepsilon, Y_2, CL$ and $\theta$ in this model should be estimated by section model wind tunnel testing. Several methods for estimating the parameters were outlined by Scanlan (1981) Ehsan (1990) and Li (1995) etc. The other notations in Eq.19 denote the same means as Eq.1.

At lock-in, it is assumed that the parameters of aerodynamic stiffness term $Y_2$ and instantaneous lift coefficient $CL$ are very small compared to aerodynamic damping term $Y_1$. It means $Y_2=CL=0$. Although an elastic suspended section model is assumed to behave in a two dimensional manner, it is actually a three-dimensional object so that the aerodynamic force are not perfectly correlated along its length. In the case of an actual bridge, the vortex-induced aerodynamic force introduced by vortices is imperfect correlation along the span. Thus $Y_1$ and $\varepsilon$ estimated from section model testing actually represent these forces in an average sense only, they may be varied with the location $x$ along the span. Thus, the modified model can be expressed as:

$$m(\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2y) = \rho UD\left[ Y_1(K,x)\left(1 - \varepsilon(K,x)\frac{y^2}{D^2}\right)\frac{\ddot{y}}{U} + Y_2(K)\frac{y}{D} + \frac{1}{2} C_L \sin(\alpha + \theta) \right]$$

(19)

The non-dimensional form of the model should be written in the following form by non-dimensional process and introducing generalized coordination $\xi(t)$:

$$m(\ddot{\xi} + 2\zeta\omega_0\dot{\xi} + \omega_0^2\xi) = \rho UD\left[ Y_1(K,x)(1 - \varepsilon(K,x)\phi^2(x)\xi^2(t))\phi(x)\dot{\xi}\right]$$

(21)

Multiplying both sides by $\phi(x)$ and integrating over the span $L$ of the deck:

$$M(\ddot{\xi}(t) + 2\zeta\omega_0\dot{\xi}(t) + \omega_0^2\xi(t)) = \rho UD\int_0^L Y_1(K,x)(1 - \varepsilon(K,x)\phi^2(x)\xi^2(t))\phi^2(x)\dot{\xi}(t)dx \quad (22)$$

Where $M$ is the generalized mass, $M = \int_{-L/2}^{L/2} m(x)\phi(x)^2 dx$.

3.2 Section model case

For section model case, the mode shape function is $\phi(x)=1$, the reduced factor varies with length of model and amplitude. The mathematic model can be written as following by inserting $\phi(x)=1$ into Eq.22.

$$M(\ddot{\xi}(t) + 2\zeta\omega_0\dot{\xi}(t) + \omega_0^2\xi(t)) = \rho UDLF(t) \quad (23)$$

Here, $F(t) = \frac{1}{L} \int_{-L/2}^{L/2} f(t,x)dx$ , $f(t,x) = Y_1(K,x)(1 - \varepsilon(K,x)\xi^2(t))\dot{\xi}(t)$.

The correlation of $F(t)$ is studied by using the similar method as last chapter (Eqs.5-9). The reduced relation of vortex induced aerodynamic forces between two-dimensional and three-dimensional is still $F=\Phi \cdot f$. Reduced factor $\Phi$ of vortex induced response between two-dimensional and three-dimensional is defined as follow:

$$\Phi = \frac{1}{L^2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} R(\Delta x)dx_1 dx_2 \quad (24)$$
Eq. 23 can be rewritten as the following form by using correlation function proposed by Wilkinson,

\[ M \left\{ \ddot{\xi}(t) + 2\zeta \omega_0 \dot{\xi}(t) + \omega_0^2 \xi(t) \right\} = \Phi \rho UDL \left[ \frac{1}{2} \dot{\varepsilon}^2 \xi(t) \right] \dot{\xi}(t) \quad (25) \]

It is worth mentioned that the parameters of \( Y_1' \) and \( \varepsilon' \) in this equation are based on absolute two-dimensional estimation. It may be quiet different with the one estimated from section model wind tunnel testing. After analyzing the right side of Eq. 22, it is clearly that the parameter \( \varepsilon \) is constant and retains its two-dimensional value and parameter \( Y_1 \) has the same reduced factor with vortex induced force. It means that,

\[ Y_1' = Y_1 / \Phi \quad \varepsilon' = \varepsilon \quad (26) \]

If the net energy loss or gain per cycle is zero at lock-in,

\[ \int_0^T \left[ 2M\zeta \omega_0 - \Phi \rho UDL Y_1 \left( 1 - \varepsilon^2 \xi(t) \right) \right] \dot{\xi}^2(t) dt = 0 \quad (27) \]

Assuming that the generalized coordinate is sinusoidal, i.e. \( \xi(t) = \xi_0 \cos(\omega t) \), then,

\[ \ddot{\xi}(t) = \dot{\xi}_0 \cos(\omega t) ; \quad \int_0^T \dot{\xi}^2(t) dt = \pi \omega \xi_0^2 ; \quad \int_0^T \dot{\xi}^2(t) \dot{\xi}^2(t) dt = \frac{\pi}{4} \omega \xi_0^4 \]

The energy condition leads to the following expression for the amplitude of the generalized coordinate \( \xi_0 \),

\[ \xi_0 = 2 \sqrt{\frac{\Phi Y_1' - 2M\zeta \omega_0}{\rho UDL \Phi Y_1' \varepsilon' \varepsilon}} \quad (28) \]

In which, \( M = mL \). Let \( m_r = \rho D^2/m \), Eq. 28 can be rewritten as:

\[ \xi_0 = 2 \sqrt{\frac{1}{\varepsilon' \left( 1 - \frac{2\zeta K}{m_r \Phi Y_1'} \right)^{1/2}}} \quad (29) \]

3.3 Prototype bridge

In the case of an actual bridge, it has an infinite set of modes. The mode shape function \( \phi(x) \) of the bridge should be gained by mode analysis. For arbitrary \( \phi(x) \), the vertical displacement at lock-in can be calculated using the condition that net energy loss or gain per cycle is zero. The energy condition leads to the following expression for the amplitude of the generalized coordinate \( \xi_0 \):

\[ \xi_0 = 2 \sqrt{\frac{\int_0^L Y_1' \phi^2(x) \frac{dx}{L} - \frac{2M\zeta K}{\rho D^2 L}}{\int_0^L Y_1' \varepsilon \phi^4(x) \frac{dx}{L}}} \quad (30) \]

Ehsan (1990) proposed a roughly method to account for the imperfect correlation of aerodynamic forces along the span. The similar method with linear theory (Chapter 2) and Duhamel Integral theory are applied in this study. Eq. 22 can be rewritten as follow:
According to Duhamel Integral theory, mode shape function $\phi(x)$ can be seen as input function, and correlation function $R(\Delta x)$ can be treated as response functions. The mathematic model can be written as following form:

$$M(\ddot{\xi}(t) + 2\zeta\omega_0\dot{\xi}(t) + \omega_0^2\xi(t)) = \rho UDL \left( Y^1_1\Phi_2 - Y_1^e'\Phi_4^2 \right) \dot{\xi}(t)$$  \hspace{1cm} (31)

Where

$$\Phi_2 = \frac{1}{L^2} \int_{L/2}^{L/2} \int_{L/2}^{L/2} \phi^1(x_1)\phi^1(x_2)R(\Delta x)dx_1dx_2$$

$$\Phi_4 = \frac{1}{L^2} \int_{L/2}^{L/2} \int_{L/2}^{L/2} \phi^1(x_1)\phi^1(x_2)R(\Delta x)dx_1dx_2$$  \hspace{1cm} (32)

The energy condition leads to the following expression for the amplitude of the generalized coordinate $\xi_0$:

$$\xi_0 = \sqrt{\Phi_2 Y_1' - \frac{2M\zeta\omega_0}{\rho UDL}}$$  \hspace{1cm} (33)

$$\xi_0 = \sqrt{\frac{\Phi_2}{\Phi_4} \left( 1 - \frac{2\zeta K}{m_r \Phi_2 Y_1'} \int_0^L \phi^2(x)dx \right)^{1/2}}$$  \hspace{1cm} (34)

Eq.34 can be re written as,

$$\xi_0 = \sqrt{\frac{\Phi_2}{\Phi_4} \left( 1 - \frac{2\zeta K}{m_r \Phi_2 Y_1'} \int_0^L \phi^2(x)dx \right)^{1/2}}$$  \hspace{1cm} (35)

If the parameters of the model indentified from sectional model testing are available, the vortex induced response of bridge can be estimated by using Eq.35 and taking into account the imperfect spanwise correlation of aerodynamic forces.

### 3.4 Application of nonlinear method

For section model, $\phi(x) = 1$, $\theta(\Delta x) = 2(L - \Delta x)$, and the reduced factor can be obtain by inserting it into Eq.24:

$$\Phi = \sqrt{\int_0^L \left( 2(L - x)R(\Delta x) \right) d\Delta x}$$  \hspace{1cm} (36)

Lets $z = \Delta x / L$ and $\hat{L} = L / D$, the non-dimensional form of Eq.36 can be expressed as follow:

$$\Phi(\hat{L}) = \sqrt{\int_0^1 (1-z) \exp[-f_1(\eta) - \hat{L}z f_1(\eta)] dz}$$  \hspace{1cm} (37)

For prototype or full bridge model of long span suspension bridge, the mode shape function of first symmetric vertical can be assuming as the form $\phi(x) = \cos(\pi \cdot x / L)$. The reduced factor can be obtained by inserting it into Eq.33:
\[
\Phi_2 = \sqrt{\frac{1}{L^2} \int_{-L/2}^{L/2} \cos^2 \left( \frac{x}{L} \right) \cdot \cos^2 \left( \frac{x + \Delta x}{L} \right) dx \cdot \exp \left[ -f_1(\eta) \cdot (\Delta x / D) f_z(\eta) \right] d\Delta x}
\]

\[
\Phi_4 = \sqrt{\frac{1}{L^2} \int_{-L/2}^{L/2} \cos^2 \left( \frac{x}{L} \right) \cdot \cos^2 \left( \frac{x + \Delta x}{L} \right) dx \cdot \exp \left[ -f_1(\eta) \cdot (\Delta x / D) f_z(\eta) \right] d\Delta x}
\]

Let \( \hat{x} = x / L \), the non-dimensional form of Eq.38 can be written as:

\[
\Phi_2 = \sqrt{2 \int_{-1/2}^{1/2} \cos^2 (\hat{x}) \cdot \cos^2 (\hat{x} + z) d\hat{x} \cdot \exp \left[ -f_1(\eta) \cdot (\hat{z}) f_z(\eta) \right] d\hat{z}}
\]

\[
\Phi_4 = \sqrt{2 \int_{-1/2}^{1/2} \cos^2 (\hat{x}) \cdot \cos^2 (\hat{x} + z) d\hat{x} \cdot \exp \left[ -f_1(\eta) \cdot (\hat{z}) f_z(\eta) \right] d\hat{z}}
\]

4 CASE STUDY

Obvious vortex induced vibration (VIV) was observed during sectional model testing and prototype for the Deer Isle-Sedgwick bridge in the third vertical mode. Details of the testing are given by Ehsan (1988) and Kumarasena (1989). The estimated parameter values at \( K=0.929 \) and \( \zeta=0.00625 \) were \( Y_1=10.54 \) and \( e=228.7 \). The main span of the bridge is 329.2m long; the deck is 1.98m deep. The mass ratio was computed to be \( m_r=0.001348 \). The peak-to-peak amplitude derived from section model testing is interpreted into prototype bridge with linear VIV theory and nonlinear VIV theory are 0.076m and 0.101m respectively by considering imperfect correlation of vortex-induced aerodynamic force along span. The amplitude observed in the field is 0.096m. It’s obvious that the proposed response estimation method yields very reasonable results.

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6 REFERENCES