Subject headings: suspension bridge, resonance, self-excited vertical vibrations.

ABSTRACT: The possibility of large oscillations of a bridge deck caused by cross winds of relatively low speed is one of the problems in dynamics of suspension bridges. The above-mentioned phenomenon is treated as vortex-induced oscillations. The purpose of this paper was to develop a semi-empirical mathematical model describing the dynamic behavior of the suspension bridge and determination of its maximum amplitude at vertical vibrations caused by vortex shedding. Taking into account the need to account for the interaction between the flow of wind and streamlined bridge, the model was developed on the basis of a combination of two approaches: the analysis of dynamics of a suspension bridge (a linear model) under external uniformly distributed along the bridge harmonic vertical load

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and the Hartlen & Currie model describing the interaction between a smooth flow and streamlined cylinder where the latter can strongly affect the lift force. Based on the joint consideration of both approaches, the extended Hartlen & Currie model was obtained to describe the behavior of the suspension bridge under the condition of vortex shedding. This model permits two regimes of self-exciting vibrations. One of them is the case, when the ratio between a Strouhal’s frequency and one of the natural frequencies corresponding with the vertical mode of the bridge is close to 1:1, and another, when the same ratio is close to 2:1. Thus, the suspension bridge can have two critical speeds for each of its natural frequency. In accordance with the proposed model, if a higher speed is achievable, then the full aeroelastic model of the bridge should be tested in a smooth flow, although usually a sectional model for testing is considered as valid one. The response of a nonlinear two degree-of-freedom system was investigated. The perturbation method of multiple time scales was used to construct first-order nonlinear differential equations and to determine steady state solutions and their stability. The bifurcation diagrams were plotted for both cases. Suppressing of undesirable vibrations of the bridge deck was discussed. Numerical calculations showed that decreasing the lift force by installing fairings or increasing damping of the suspension bridge or their combination could decrease vertical self-excited vibrations or even prevent their arising.

Introduction

The design and construction of the long span suspension bridges in the last decades are accompanied by intensive investigations of such bridges for instability under wind actions.
The long suspension bridges are bluff bodies by aerodynamic standards and cause the flow to become separated resulting in a wake characterized by the shedding vortices behind the bodies. These vortices are the reason of some dangerous phenomena, which can make bridges vibrate and lead to the bridge instability. They are a lot of investigations associated with the phenomenon of flutter, but comparatively few ones have been done with separated vertical or torsional vibrations induced by wind. This paper is limited to the case of across flow vibrations. Ehsan and Scanlan (1990) wrote, that if across flow and torsional frequencies were well separated, then across flow or torsional vibrations may occur in a single mode of a bridge even if both degrees of freedom were unrestrained. They noticed that for describing the above-mention phenomenon, the Hartlen&Currie model (1970) could be used. It is worth noting that majority of mathematical models describing the behavior of structures under wind actions are semi-empirical, i.e. some model parameters are obtained from the wind tunnel tests on section or full-scale models. The Hartlen&Currie (1970) nonlinear model consists of two coupled oscillators: one of them describes the behavior of a structure mode, and another one (Raileigh oscillator) does the shedding process in a vortex-structure interaction. Besides Scanlan and coauthors proposed a few additional models for bridges. The first of them (Simiu and Scanlan (1996)) is a linear model. which describes the linear oscillator (a single mode of a bridge) with aerodynamic excitation. The last is a sum of three parts depending on a dimensionless velocity, vertical displacement, and harmonic member. It has 4 parameters to be determined. The alternative, nonlinear model describes the oscillator in the van der Pol form. The next model (Elsan and Scanlan (1990) describes the behavior
of the bridge by the nonlinear van der Pol oscillator with the additional forced members represented by a dimensionless displacement and harmonic function. The last model (Scanlan (1998) is a linear oscillator with flutter derivatives. The method of flutter derivatives does not aim at explaining the phenomena involved; instead it provides an analytical framework that accommodates aeroelastic facts pertinent to necessary design dynamic calculations. While useful flutter derivative method is linear, vortex excitation is recognizable as a distinctly nonlinear phenomenon. It is should be noted that all of the above methods do not distinguish between the suspension bridges and other types of flexible bridges, such as girder or cable-stayed ones. However, suspension bridges have their own particular features. Gol’denblat (1947) has shown that the vertical harmonic load distributed along the span can cause primary or parametric resonances. Therefore, to describe the motion of a suspension bridge in the laminar or low-turbulence incident flow, a mathematical model of the one with an applied vertical harmonic load and the Hartlen&Currie model are considered together, and the result is displayed in the extended Hartlen&Currie model describing two possible resonance regimes. The first regime appears in the case where the ratio between a vortex-shedding frequency and a vertical natural frequency is close to 1:1, the second one is possible where the same ratio is close to 2:1. To solve the corresponding nonlinear differential equations, the method of multiple time scales is used. This method allows constructing the first order autonomous nonlinear differential equations and determining the steady state solutions and their stability. Suppressing of undesirable vibrations is discussed.
Mathematical model.

Basic assumptions. Basic assumptions of the forced vibrations of suspension bridges are accepted the same as in the case of free vibrations of the ones which was analyzed by Abdel-Ghaffar (1976). It is assumed that the hangers are considered to be inextensible, the dead-load curve of the cable forms a parabola, and the deformations of bridges are small. The vertical stiffness, and the dead-load of stiffening girders (trusses) and outer dynamic loads are assumed constant throughout the span. All stresses in the bridge follow Hooke’s law. The tower cable saddles are free to move horizontally upon roller nests.

Vertical vibration. For the sake of simplicity it will be examined the vertical vibrations of the one span suspension bridge (Fig.1), neglecting the mass of a cable. The assumption of the inextensible hangers means that vertical displacements of the cable and girder are equal to \( y \).

The bending moment in a girder of a suspension bridge can be calculated by the formula

\[
M = M_0 + F_x(H + y) \tag{1}
\]

where

- \( M_0 \) - the bending moment in a hinged-hinged girder (separately from a suspension bridge),
- \( F_x \) - the horizontal component of a total cable response
- \( y(x,t) \) - the deflection of the girder,
- \( H(x, t) \) - the ordinate of a cable.
In this case motion of the suspension bridge is governed by the following differential equation:

\[ m \frac{\partial^2 y}{\partial t^2} + EJ \frac{\partial^4 y}{\partial x^4} - F_x(t) \frac{\partial^2 (y + H)}{\partial x^2} = W + W(x, t) \]  

(2)

where

\( m \) - the mass of a deck per unit length of the span,

\( W = mg \),

\( \bar{W}(x, t) \) - the outer load per unit length of the span,

\( E \) - the modulus of elasticity of the both stiffening girders (trusses),

\( J \) - the area moment of inertia of the both stiffening girders (trusses),

\( L \) - the length of the span,

\( t \) - time.

The value of \( F_x(t) \) depends on the shape of a cable and its extension at the given moment of time, and represents some functional of \( y \). So the equation (2) is a complex nonlinear one (Abdel-Ghaffar (1976)). In the case linear theory Gol’denblat’s (1947) suggested that horizontal components of cable tensions, \( F_{xw}, F_{xi}(t) \) and \( F_{xd}(t) \) due to dead loads, inertial forces and external dynamic load respectively, can be estimated separately, and the horizontal component of cable tensions \( (F_{xct}) \) from a constant load can be calculated by the method of structural mechanics for statically indeterminate systems by the formula

\[ F_{xct} = \frac{\int_0^L M_1 M_2 dx}{EJ} \frac{\int_0^L M_2^2 dx}{EJ} + \sum_{i=1}^{n} \frac{N_i^2 l_i}{EF} \]  

(3)

The above formula is correct for infinitely small extensions of the cable. For finite extensions of the cable the correctness of Eq.(3) should be examined. The horizontal component of
cable tensions \( F_{xi}(t) \) can be estimated separately for antisymmetric and symmetric modes. In the case of free vibrations of antisymmetric modes \( F_{xi}(t) \) is zero.

As the consequence, the differential equation of free antisymmetric vibrations will be the following:

\[
m \frac{\partial^2 y}{\partial t^2} + EJ \frac{\partial^4 y}{\partial x^4} - F_{xw} \frac{\partial^2 y}{\partial x^2} - F_{xw} \frac{\partial^2 H}{\partial x^2} = mg
\]  

(4)

In the case of free vibrations of symmetric modes, \( F_{xi}(t) \) is not 0, but very small compared with \( F_{xw} \) and it can be neglected. Abdel-Ghaffar (1976) has showed that the actual extension for the symmetric modes higher than first one is very small and, consequently, \( F_{xi}(t) \) is also very small. It can be noted that the free vibration for the first mode is possible only if a strain of the cable is considerable (the correspondent frequency is big) and it is not studied in this paper.

Thus, Eq.(4) describes free vibration of the bridge for symmetric and antisymmetric modes.

Let us consider the case of the dynamical load which is equal to \( W_0 \cos(\Omega t) \), where \( W_0 \) is the uniform external load per unit length of the span. The dynamic horizontal component of a cable response caused by the dynamic load can be determined as

\[
F_{xd} = \alpha W_0 \cos(\Omega t)
\]  

(5)

where \( \alpha \) - some coefficient with the dimension of length (for example, m), and \( W_0 \) is determined by (3). It allows to model motion of a suspension bridge (taking in account damping) as a linear damped dynamical system:

\[
m \frac{\partial^2 y}{\partial t^2} + EJ \frac{\partial^4 y}{\partial x^4} - (F_{xw} + \alpha W_0 \cos(\Omega t)) \frac{\partial^2 y}{\partial x^2} + \Xi (\frac{\partial y}{\partial t})
\]

\[-(F_{xw} + \alpha W_0 \cos(\Omega t)) \frac{\partial^2 H}{\partial x^2} = W + W_0 \cos(\Omega t)
\]  

(6)
where

$\Xi$ - the damping operator,

$F_{xw}$ - the horizontal component of a cable response from $W$.

Eq. (6) can be solved by using the following transformation:

$$y(x, t) = \sum_{n=1}^{N} p_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

(7)

In accordance with the assumptions, the shape of a cable is changed by the law of a quadratic parabola:

$$H = 4f x(L - x)/L^2$$

(8)

where $f$ - the sag of a cable (the cable deflection at the mid-span ($x=L/2$)).

It results in

$$\frac{\partial^2 H}{\partial t^2} = -\frac{8f}{L^2}$$

(9)

Substituting Eq. (7) and (9) into Eq. (6), neglecting constant members, and under the hypothesis of classical damping, the equation of motion of the $n$th principal coordinate is expressed as

$$\ddot{p}_n(t) + 2\zeta_n \omega_n \dot{p}_n(t) + \omega_n^2 (1 + \gamma_n \cos(\Omega t)) p_n(t) = \overline{a_n} b W_0 \cos(\Omega t)/m$$

(10)

where

$$\overline{b} = (1 - \frac{8F_{xw}f}{WL^2}),$$

$$\overline{a_n} = \frac{4}{n\pi} \text{ for } n = 1, 3, 5..., $$

$$\overline{a_n} = 0 \text{ for } n = 2, 4, 6..., $$
\[ \zeta_n - \text{a damping coefficient}, \]

\[ \omega_n^2 = \frac{F_{xw} n^2 \pi^2}{m L^2} c_n \quad (11) \]

\[ \gamma_n = \frac{W_0}{W c_n} \quad (12) \]

\[ c_n = \frac{1}{(1 + \frac{EJn^2 \pi^2}{F_{xw} L^2})} \quad (13) \]

where \( n = 1, 2, 3 \ldots \)

Eq. (10) describes the parametric excitation of a single-degree-of freedom and practically coincide to the classic Mathieu equation (e.g. Nayfeh and Mook (1995)). Depending on the values of the parameters, the solution (10) can be stable, unstable or periodical.

The formula (12) straightforward gives some results which is known from the engineering experience:

1. decreasing \( W \) (the light suspension bridges) promotes arising instability of suspension bridges;
2. increasing \( W_0 \) is correspondent with increase of streamlining of bridges;

**Vertical vortex-induced vibrations.** The further aim of this paper is to examine the vibratory response of suspension bridges by the phenomenon of vortex-excited vibration. Analytically the problem is very difficult, involving a periodic separated flow around an oscillatory boundary. To simplify the problem, the Hartlen - Currie model is used and extended in order to apply it for the results got above. The pointed model is semi-empirical because the part of its coefficients should be extracted experimentally. The important feature of the model is that it consider an interaction between the flow and a bridge because the
latter can strongly affect the lift force of the flow. Hartlen and Currie (1970) treated the fluctuating lift force (associated with vortex shedding) as the one arising from some oscillator in the separated flow. These types of models are known as wake oscillators models. The idea first was suggested by Bishop and Hassan (1964).

Treating the external force $W_0 \cos(\Omega t)$ as the lift force of the flow into Eq.(10), and assuming perfect spanwise correlation of the flow, the following equation can be obtained:

$$\ddot{p}_n(t) + \omega_n^2 p_n(t) = -2\zeta_n \omega_n \dot{p}_n(t) - 0.5\frac{\rho V^2 b}{W} c_n p_n(t) + 0.5\frac{\pi_n}{m} \rho V^2 b c_l$$

where

$\rho$ - the air density,

$V$ - the flow velocity,

$c_l$ - the instantaneous lift coefficient.

Taking into consideration the Strouhal relation

$$V = \frac{f_s h}{S}$$

where

$f_s$ - the vortex shedding frequency,

$h$ - the height of a deck,

$S$ - the Strouhal number,

designating $\tau = \omega_n t$, $p_r = p_n(t)/b$, $\omega_n = 2\pi f_n$, $\omega = f_s/f_n$, $\omega_1 = 1$, $\zeta_n = \zeta$ and using this designation into Eq.(14), the latter becomes

$$p''_r + \omega_1^2 p_r = -2\zeta p'_r - a_1 \omega^2 c_l p_r + a_2 \omega^2 c_l$$

(16)
where \((\cdot)'' = d^2(\cdot)/d\tau^2\), \((\cdot)' = d(\cdot)/d\tau\),

\[
a_1 = \frac{1}{8} \frac{\rho h^2 n^2 F_{xw} b}{L^2 W m}
\]

\[
a_2 = \frac{1}{8} \frac{\rho h^2 \sigma_n b}{\pi^2 m}
\]

(17)

(18)

The equation for \(c_l\) is similar to the one in the Hartlen and Currie model, in which \(c_l\) is derived from an equivalent oscillator embodying the primary characteristics deduced from experimental evidence. The form of the equation for \(c_l\) will be:

\[
\ddot{c}_l + (\text{damping term}) + \omega^2 c_l = (\text{forcing term})
\]

(19)

where \((\omega_2 = 2\pi f_s)\) assures agreement with the Strouhal relation. Rewriting Eq.(25) in the time scale gives:

\[
c_l'' + (\text{damping term}) + \omega^2 c_l = (\text{forcing term})
\]

(20)

The damping term must be such that the oscillator is self-excited and self-limited. So it is comfortable to take Rayleigh (or van der Pol) oscillator as a damping term. In this case the equation for \(c_l\) becomes:

\[
c_l'' - \alpha_1 c_l' + \gamma_1 (c_l')^3 + \omega^2 c_l = (\text{forcing term})
\]

(21)

where the limit amplitude for small \(\alpha_1\) is

\[
c_{lo} = \sqrt{\frac{4\alpha_1}{3\gamma_1\omega^2}}
\]

(22)

Putting \(\alpha_1 = \bar{\sigma}\omega\) and \(\gamma_1 = \gamma/\omega\), the equation for \(c_l\) becomes

\[
c_l'' - \bar{\sigma}\omega c_l' + \frac{\gamma}{\omega} (c_l')^3 + \omega^2 c_l = (\text{forcing term})
\]

(23)
According to the Hartlen and Currie model, the forcing term is taken, rather arbitrarily, to be proportional to the vertical velocity. In the case of a considered problem, the Hartlen and Currie model should be extended by the additional quadratic term, and the equation (23) becomes

\[ c''_l - \alpha \omega c'_l + \frac{\gamma}{\omega} (c'_l)^3 + \omega^2 c_l = b_1 (x'_r)^2 + b_2 x'_r \]  

(24)

where \( b_1 \) and \( b_2 \) - the constants determined by the experiment. Putting \( p_r = z_1 \) and \( c_l = z_2 \), the proposed mathematical model will be the following:

\[ z''_1 + \omega^2 z_1 = -2 \zeta z'_1 - a_1 \omega^2 z_2 z_1 + a_2 \omega^2 z_2 \]  

(25)

\[ z''_2 - \alpha \omega z'_2 + \frac{\gamma}{\omega} (z'_2)^3 + \omega^2 z_2 = b_1 (z'_1)^2 + b_2 z'_2 \]  

(26)

This model differs from the Hartlen and Currie model through the quadratic terms in the right part of the equations.

**Solution by Perturbation Method of Multiple Time scales.**

To apply the method of multiple time scales to the equations (25), (26), a rescaling similar to the one performed by Ng et al. (2001) must first take place: \( \bar{\omega} = \varepsilon \), \( 2 \zeta = \varepsilon \xi \), \( a_1 = \varepsilon \hat{a}_1 / Z_0 \), \( a_2 = \varepsilon \hat{a}_2 / Z_0 \), \( b_1 = \varepsilon Z_0 \hat{b}_1 \), \( b_2 = \varepsilon Z_0 \hat{b}_2 \), \( \gamma = \varepsilon^4 \frac{1}{Z_0^2} \),

where \( \varepsilon \) is considered as a small parameter.

In this case the equations (25), (26) will be obtained as follows:

\[ z''_1 + \omega^2 z_1 = \varepsilon (\xi z'_1 - \hat{a}_1 \omega^2 z_2 z_1 / Z_0 + \hat{a}_2 \omega^2 z_2) \]  

(27)

\[ z''_2 + \omega^2 z_2 = \varepsilon (\omega z'_2 - \frac{4}{3} \frac{1}{\omega Z_0^2} (z'_2)^3 + Z_0 \hat{b}_1 (z'_1)^2 + Z_0 \hat{b}_2 z'_1) \]  

(28)
Solutions for \( z_1 \) and \( z_2 \) are assumed in the following perturbation series with two different time scales (Nayfeh and Mook (1995)):

\[
\begin{align*}
  z_1(\tau) &= z_{10}(\tau_0, \tau_1) + \varepsilon z_{11}(\tau_0, \tau_1) + \ldots \quad (29) \\
  z_2(\tau) &= Z_0(\tau_0, \tau_1) + \varepsilon z_{21}(\tau_0, \tau_1) + \ldots \quad (30)
\end{align*}
\]

where \( \tau_n = \varepsilon^n t, \ n = 0, 1 \).

For compactness of the equations to follow, two different operators defined as

\[
\begin{align*}
  D_0 &\equiv \partial()/\partial \tau_0 \\
  D_1 &\equiv \partial()/\partial \tau_1
\end{align*}
\]

**The case of \( 2\omega_1 \) is near \( \omega \).**

Substitution of equations (29), (30) into equations (27), (28) gives the two systems of equations.

Order \( \varepsilon^0 \)

\[
\begin{align*}
  D_0^2 z_{10} + \omega_1^2 z_{10} &= 0 \quad (33) \\
  D_0^2 z_{20} + \omega^2 z_{20} &= 0 \quad (34)
\end{align*}
\]

Order \( \varepsilon \)

\[
\begin{align*}
  D_0^2 z_{11} + \omega_1^2 z_{11} &= -2D_0 D_1 z_{10} - \xi D_0 z_{10} - \dot{a}_1 \omega^2 z_{10} z_{20} + \dot{a}_2 \omega^2 z_{20} \quad (35) \\
  D_0^2 z_{21} + \omega^2 z_{21} &= -2D_0 D_1 z_{20} + \omega D_0 z_{20} - \frac{4}{3\omega} (D_0 z_{20})^3 + \ddot{b}_1 (D_0 z_{10})^2 + \ddot{b}_2 D_0 z_{10} \quad (36)
\end{align*}
\]

The steady-state solutions of Eqs. (33) and (34) can be obtained in the following form:

\[
  z_{10} = A_1(\tau_1) \exp(i\omega_1 \tau_0) + \overline{A}_1(\tau_1) \exp(-i\omega_1 \tau_0) \quad (37)
\]
\[ z_{20} = A_2(\tau_1)exp(i\omega \tau_0) + A_2^*(\tau_1)exp(-i\omega \tau_0) \]  
\[ \text{where } A_1 \text{ and } A_2 \text{ are arbitrary functions at this level of approximation. The resonance condition is considered as} \]
\[ \omega = 2\omega_1 + \varepsilon \sigma_1 \]  
where \( \sigma_1 \) is a measure of detuning off 2:1 resonance.

Substituting equations (37), (38) into equations (35), (36), dropping the hats on terms, and selecting the secular terms for the case \( \omega \sim 2\omega_1 \), the following solvability conditions will be obtained:

\[ i2A_1' \omega_1 + i\xi \omega_1 A_1 + a_1 \omega^2 A_2 \exp(i\sigma_1 \tau_1) = 0 \]  
\[ -i2A_2' \omega + i\omega^2 A_2 - i4\omega^2 A_2^2 A_2 - b_1 \omega^2 A_2^2 \exp(-i\sigma_1 \tau_1) = 0 \]  

Putting \( A_n = 0.5d_n \exp(i\beta_n) \), we obtain:

\[ d_1' = -0.5\xi d_1 + 0.25a_1 \omega^2 d_1 d_2 \sin(\gamma) \]  
\[ d_2' = 0.5\omega d_2 - 0.5\omega d_2^3 - 0.25b_1 \frac{d_1^2}{\omega} \sin(\gamma) \]  
\[ \gamma' = -\sigma_1 + 0.5(a_1 \omega^2 d_2 - 0.5b_1 \frac{d_1^2}{\omega d_2} \cos(\gamma)) \]  

where

\[ \gamma = 2\beta_1 - \beta_2 - \sigma_1 \tau_1 \]  

The case of \( \omega_1 \) is near \( \omega_1 \) .
The resonance condition is considered as

\[ \omega^2 = 1 + \varepsilon \sigma_2 \]  \hspace{1cm} (46)

where \( \sigma_1 \) is a measure of detuning off 1:1 resonance, and \( \omega_1 = 1 \).

Substituting equations (29), (30) into equations (27), (28), and taking into considerations the resonant condition (46), the following system of equations can be obtained:

**Order \( \varepsilon^0 \)**

\[ D_0^2 z_{10} + z_{10} = 0 \]  \hspace{1cm} (47)

\[ D_0^2 z_{20} + z_{20} = 0 \]  \hspace{1cm} (48)

**Order \( \varepsilon \)**

\[ D_0^2 z_{11} + \omega_1^2 z_{11} = -2D_0D_1z_{10} - \xi D_0z_{10} - \hat{a}_1z_{10}z_{20} + \hat{a}_2z_{20} \]  \hspace{1cm} (49)

\[ D_0^2 z_{21} + \omega_1^2 z_{21} = -\sigma_2 z_{20} - 2D_0D_1z_{20} + D_0z_{20} - \frac{4}{3}(D_0z_{20})^3 + \hat{b}_1(D_0z_{10})^2 + \hat{b}_2D_0z_{10} \]  \hspace{1cm} (50)

The steady-state solutions of Eqs. (47) and (48) can be obtained in the following form:

\[ z_{10} = A_1(\tau_1)e^{i\tau_0} + \overline{A_1}(\tau_1)e^{-i\tau_0} \]  \hspace{1cm} (51)

\[ z_{20} = A_2(\tau_1)e^{i\tau_0} + \overline{A_2}(\tau_1)e^{-i\tau_0} \]  \hspace{1cm} (52)

Substituting equations (51), (52) into equations (49), (50), dropping the hats on terms, and selecting the secular terms for the case \( \omega \sim \omega_1 \), the following solvability conditions will be obtained:

\[ -i2A_1' - i\xi A_1 + \sigma_1 A_2 = 0 \]  \hspace{1cm} (53)

\[ -i2A_2' + iA_2 - \sigma_2 A_2 - i4A_2A_2 + i\xi A_1 = 0 \]  \hspace{1cm} (54)
Putting $A_n = 0.5\overline{d}_n \exp(i\beta_n)$, we obtain:

\begin{align}
\overline{d}_1' &= -0.5\xi\overline{d}_1 - 0.5a_2\overline{d}_2 \sin(\gamma) \\
\overline{d}_2' &= 0.5\overline{d}_2 - 0.5\overline{d}_2^3 + 0.5b_2\overline{d}_1 \cos(\gamma) \\
\gamma' &= -0.5(\sigma_2 + a_2 \frac{\overline{d}_2}{\overline{d}_1} \cos(\gamma) + b_2 \frac{\overline{d}_1}{\overline{d}_2} \sin(\gamma))
\end{align}

where

\begin{equation}
\gamma = \beta_1 - \beta_2
\end{equation}

**Numerical results.**

To illustrate the behavior of the model, two cases are considered. For the first regime (the frequency ratio 1:1), Fig. 2 shows the result of numerically integrating equations (55)-(57) by using parameters in the study by Hartlen and Currie (1970) ($a_2 = 0.002$, $b_2 = 0.4$, $\xi = 0.0015$). The large amplitudes of the resonance region occur when the vortex-shedding frequency is near a natural frequency of the system. Fig. 2 is the bifurcation diagram in which the vertical ($\overline{d}_1$) amplitude is plotted as a function of the control parameter $\sigma_2$ (detuning). Fig. 2 exhibits bistability i.e. depending on initial conditions, at a given control parameter the system might develop to more than one stable state. Exsan (Scanlan (1998)) experimentally studied the four bluff section models: 4:1 H - section, 4:1 solid rectangle, Deep Isle Bridge, Original Tacoma Narrows Bridge. All of the result responses resemble each other and Fig. 2 but the detail nature of the responses are differed. It is worth noting that the system of equations (55)-(57) can be used for description of dynamical behavior of cable-stayed and
girder bridges, and stacks under wind action. For the second regime (the frequency ratio 2:1) Fig. 3 shows the result of numerically integrating equations (42)-(44) by using parameters \( a_1 = 0.008, b_1 = 0.4, \xi = 0.0015 \). The vertical mode loses its stability in the vicinity of \( \sigma_1 = 0 \). The process also exhibits bistability and has symmetry concerning \( \sigma_1 = 0 \). In this case the maximum of the amplitude \( d_1 \) is much more in comparison for the first case because the lift force 4 times more. The proposed model permits to produce a variety of countermeasures by which inconvenient wind-induced vibrations can be reduced to allowable levels of motion. One of these is installation of passive tuned mass dampers or dampers for increasing damping of a bridge. Moreover, calculations performed for both regimes (Fig. 2 and Fig. 3) showed that for \( \xi = 0.035 \) for the first case, \( \xi = 0.015 \) for the second one, self-excited vibrations did not occur. Decreasing the lift force by installing fairings to the side faces can be another countermeasure. Calculations performed for both regimes showed that self-excited vertical vibrations did not occur for \( a_1 = 0.002 \) (the second regime), and for \( a_2 = 0.0005 \) (the first one). Sometimes it is convenient to use simultaneously both the above-mentioned countermeasures. In this case the level of necessary damping will be less.

**Conclusion.**

The present paper offers the model to estimate the across-wind response of suspension bridges in a laminar flow. The modal permits two regimes of self-exciting vibrations. One of them is the case where the ratio between a vortex-shedding frequency and one of the natural vertical frequencies is close to 1:1, and another one where the same frequency is close to 2:1. Both
responses of a suspension bridge exhibit bistability, i.e. dependence on initial conditions. Thus, a suspension bridge can have two critical velocities of a laminar flow for each of its natural vertical frequency. In accordance with the proposed model, if a higher velocity achievable, then the full aeroelastic model of the bridge should be tested in a smooth flow. Numerical calculations show that decreasing the lift force by installing fairings, or increasing damping of a suspension bridge, or their combination, mitigates vertical self-exciting (vortex-induced) vibrations or even prevents their arising.

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FIGURES

Fig. 1. Suspension bridge.

Fig. 2. Bifurcation diagram. Vertical amplitude ($d_1$) response as function of a control parameter $\sigma_2$ (for $a_2 = 0.002$, $b_2 = 0.4$, $\xi = 0.0015$).

Fig. 3. Bifurcation diagram. Vertical amplitude ($d_1$) response as function of a control parameter $\sigma_1$ (for $a_2 = 0.008$, $b_2 = 0.4$, $\xi = 0.0015$).